

A generalized Neyman-Pearson lemma for two sublinear operators

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Abstract

In this paper we extended the Neyman-Pearson lemma by replacing two probabilities into two sublinear operators and divide our problem into two cases to get the reminiscent form of the optimal solution as in the linear case if the optimal solution exists. We also studied the existence of the optimal solution.

Keywords: duality; G -expectation; hypothesis testing; Mazur-Orlicz theorem; pure additive set function.

1 Introduction

For a given measurable space (Ω, \mathcal{F}) , suppose that there are two probability measures P and Q . If we want to discriminate between them. we can try to select a test function $X : \Omega \rightarrow [0, 1]$, which rejects P on ω with probability $X(\omega)$. Then $E_P[X]$ is the probability of rejecting P when it is true (Type I error) and $E_Q[1 - X]$ is the probability of accepting P when it is false (Type II error). In generally, to minimize both of the Type errors is impossible. The traditional method is to find a test, which minimizes Type II error while keeping Type I error below a given acceptable significance level $\alpha \in (0, 1)$. The most famous result is Neyman-Pearson lemma, in which it tells us the optimal test function not only exists but also satisfies some reminiscent form.

In real life the case may be more complicated. For example, there will be not only just two probability measures but two families of probability measures for us to discriminate. In 1973, Huber and Strassen [8] studied hypothesis testing problem for Choquet capacities. In 2001, Cvitanic and Karatzas [3] studied the min-max test by using convex duality method. In 2008, Ji and Zhou [10] studied hypothesis tests for g -probabilities. In 2010, Rudloff and Karatzas [15] studied composite hypothesis by using Fenchel duality. The similar problem also arises in the financial mathematics (refer Rudloff [14]).

In [3], Cvitanic and Karatzas assume \mathcal{P} and \mathcal{Q} are two families of probability measures, $\mathcal{P} \cap \mathcal{Q} = \emptyset$ and $P \ll K, Q \ll K, \forall P \in \mathcal{P}, \forall Q \in \mathcal{Q}$ for some probability measure K . Then set

$$\mathcal{G} := \{G_P := \frac{dP}{dK}; P \in \mathcal{P}\}, \quad \mathcal{H} := \{H_Q := \frac{dQ}{dK}; Q \in \mathcal{Q}\}.$$

For the sake of our statement clearly, here we assume \mathcal{G} and \mathcal{H} are closed sets in the sense of K -a.s.. Then they can find a quadruple $(\hat{G}, \hat{H}, \hat{z}, \hat{X}) \in (\mathcal{G} \times \mathcal{H} \times (0, \infty) \times \mathcal{X}_\alpha)$ such that

$$E_K[G\hat{X}] \leq E_K[\hat{G}\hat{X}] = \alpha, \quad \forall G \in \mathcal{G}.$$

$$E_K[\hat{H}X] \leq E_K[\hat{H}\hat{X}] \leq E_K[H\hat{X}], \quad \forall X \in \mathcal{X}_\alpha, \forall H \in \mathcal{H}.$$

and

$$\hat{X} = I_{\{z\hat{G} < \hat{H}\}} + B \cdot I_{\{z\hat{G} = \hat{H}\}}$$

for some suitable random variable $B : \Omega \rightarrow [0, 1]$.

In our paper, we use a new method to get the form of the optimal test function. One reason is sometimes a reference probability measure K such that all elements in \mathcal{P} and \mathcal{Q} are dominated by it will not exist. Example 1.1 is in such case. Another reason is, sometimes not all the optimal solutions attain the significance level α .

Example 1.1 Let $\Omega := [0, 1]$, \mathcal{F} is all the Borel set on $[0, 1]$ and δ_x ($0 \leq x \leq \frac{1}{2}$) be the measure on $[0, 1]$ defined as

$$\delta_x(\omega) = \begin{cases} \frac{2}{3}, & \omega = x; \\ \frac{1}{3}, & \omega = x + \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{P} := \{\delta_x, x \in [0, \frac{1}{2}]\},$$

$$\mathcal{Q} = \{Q\}, \quad Q(\omega) = \begin{cases} \frac{1}{2^{k+1}}, & \omega = \frac{1}{2^k}, k \geq 1; \\ \frac{1}{2}, & \omega = \frac{3}{4}; \\ 0, & \text{otherwise.} \end{cases}$$

We can check that there does not exist a P_0 such that $P \ll P_0, \forall P \in \mathcal{P}$. If we take $\alpha = \frac{1}{3}$, then the optimal solution X_α satisfies $E_\mu[X_\alpha] = \frac{1}{3}$.

Obviously, the optimal solution is not unique, and

$$X(\omega) = \begin{cases} \frac{1}{2}, & \omega = \frac{1}{2^k}, k \geq 1, k \neq 2; \\ 0, & \omega = \frac{1}{4}; \\ 1, & \omega = \frac{3}{4}; \\ 0, & \text{otherwise.} \end{cases}$$

is one of the optimal solutions.

In this paper, we replace the two probability measures by two sublinear operators. Our question is whether there still exists the optimal test function. Furthermore, if such an optimal solution exists, whether it has the reminiscent form as in classical Neyman-Pearson lemma.

Different from the previous work of Cvitanic and Karatzas', with the help of the Mazur-Orlicz theorem, we only need three mild assumptions besides the operators are sublinear. The cost is we lose the existence of the optimal solution. We have obtained that every optimal solution has the reminiscent form as long as

it exists. Furthermore, we study the case that the optimal solutions do not reach the significance level α in detail.

Need to point out that if we assume the two sublinear operators are both continuous from above as done in subsection 4.1, the three mild assumptions can be abandoned and there will exist a reference probability measure K such that all the additive set functions dominated by either of the two sublinear operators are absolutely continuous with respect to K . In this case, we can get the existence of the optimal solution, but our framework is still different from Cvitanic and Karatzas'. In fact, the topology in their paper both on test function space and on probability measures space can be considered as the one generated by P_0 -a.s., in contrast to the topology in our paper on test function is generated by the $L^\infty(K)$ -norm and the topology on the probability measures space is generated by the $L^1(K)$ -norm(or the *weak** topology defined on $L^1(K)$).

The paper is organized as follows. In section 2, we state our problem. In section 3, we transform the initial problem and divide into two cases to get the representation of the optimal solution. In section 4, we have discussed the existence of the optimal solution. In section 5, we study the hypothesis testing problem when the test functions are restrained to be chosen in a smaller space L_c^1 .

2 Statements of our problem

Denote \mathcal{X} as the set of all bounded measurable functions on the measurable space (Ω, \mathcal{F}) , \mathbb{N} as the set of the natural number and \mathbb{R} as the set of the real number. Recall \mathcal{X} is a Banach space endowed with the supremum norm. \mathcal{X}^* stands for the dual space of \mathcal{X} . We write $\langle X, X^* \rangle$ as the value of X^* at X and $\sigma(\mathcal{X}^*, \mathcal{X})$ as the weak* topology on \mathcal{X}^* .

Definition 2.1 *We call an operator ρ is sublinear, if it satisfies the following properties:*

- (i) *Monotonicity:* $\rho(\xi_1) \geq \rho(\xi_2)$ if $\xi_1 \geq \xi_2$.
- (ii) *Constant preserving:* $\rho(c) = c$ for $c \in \mathbb{R}$.
- (iii) *Sub-additivity:* For each $\xi_1, \xi_2 \in \mathbb{F}$, $\rho(\xi_1 + \xi_2) \leq \rho(\xi_1) + \rho(\xi_2)$.
- (iv) *Positive homogeneity:* $\rho(\lambda\xi) = \lambda\rho(\xi)$ for $\lambda \geq 0$.

Denote E_μ and E_v as the two sublinear operators to discriminate, $E_{\bar{\mu}}$ and $E_{\bar{v}}$ are their conjugation operators, i.e.

$$E_{\bar{\mu}}[X] = -E_\mu[-X], \forall X \in \mathcal{X}$$

and

$$E_{\bar{v}}[X] = -E_v[-X], \forall X \in \mathcal{X}.$$

Our aim is to select a test function $X \in [0, 1]$ to minimize the Type II error $E_v[1 - X]$ while keeping the Type I error $E_\mu[X]$ less than a significance level α . This is equivalent to solve the following problem.

Problem 2.2 For a given significance level $\alpha \in (0, 1)$, whether there exists a test function $X_\alpha \in \mathcal{X}_\alpha$ such that

$$E_{\bar{v}}[X_\alpha] = \sup_{X \in \mathcal{X}_\alpha} E_{\bar{v}}[X], \quad (2.1)$$

where \mathcal{X}_α is the set $\{X; E_\mu[X] \leq \alpha, X \in [0, 1], X \in \mathcal{X}\}$.

If such a X_α exists, we call it the optimal solution of Problem 2.2. It is easy to see that \mathcal{X}_α is a bounded convex set.

3 Characterization of the optimal solution

In this section, we will always assume the optimal solution of Problem 2.2 exists. Under this assumption, we study the necessary condition of the optimal solution and the representation of the optimal solution is obtained.

3.1 Properties of the optimal solution

Lemma 3.1 (Mazur-Orlicz theorem) Let \mathcal{Z} be a nonzero vector space, $\rho : \mathcal{Z} \rightarrow \mathcal{R}$ be sublinear and D be a nonempty convex subset of \mathcal{Z} . Then there exists a linear functional L on \mathcal{Z} such that $L(Z) \leq \rho(Z)$ for any $Z \in \mathcal{Z}$ and

$$\inf_{Z \in D} L(Z) = \inf_{Z \in D} \rho(Z).$$

Proof. Refer the Lemma 1.6 of chapter I in [16]. ■

Remark 3.2 In fact, by the proof of the lemma 1.6 of chapter I in [16], the linear operator L we want to find in Lemma 3.1 can be chosen as any one of the linear operators dominated by the sublinear operator $\hat{\rho}$ defined as:

$$\hat{\rho}(Y) := \inf_{\lambda > 0, Z \in D} [\rho(Y + \lambda Z) - \lambda \beta],$$

where $\beta := \inf_{Z \in D} \rho(Z)$.

Corollary 3.3 For any given $X \in \mathcal{X}$, there exists $L \in \mathcal{X}^*$ such that $L \leq E_\mu$ and

$$L(X) = E_\mu[X].$$

Proof. This is the direct result from Lemma 3.1. ■

Remark 3.4 By Theorem A.50 in [7], for any linear operator $L \in \mathcal{X}^*$, there exists a unique bounded finitely additive set function P on the measurable space (Ω, \mathcal{F}) such that

$$L(X) = \int X dP \quad \text{for all } X \in \mathcal{X}.$$

In order to show the relationship between the elements in \mathcal{X}^* and the additive set functions, we will denote the linear operator $L \in \mathcal{X}^*$ as E_P . Furthermore, we take the set \mathcal{P} as $\{P, E_P \leq E_\mu\}$ and \mathcal{Q} as $\{Q, E_Q \leq E_v\}$, it is easy to check \mathcal{P} and \mathcal{Q} are also convex. By the Proposition 2.85 in [7], we have

$$E_\mu[X] = \sup_{P \in \mathcal{P}} E_P[X] \quad \text{and} \quad E_v[X] = \sup_{Q \in \mathcal{Q}} E_Q[X].$$

Then

$$E_{\bar{\mu}}[X] = \inf_{P \in \mathcal{P}} E_P[X] \quad \text{and} \quad E_{\bar{v}}[X] = \inf_{Q \in \mathcal{Q}} E_Q[X].$$

In the following, without confusion, we will use the sets \mathcal{P} (resp. \mathcal{Q}) to denote either the set of elements dominated by E_μ (resp. E_v) or the set of its related additive set functions. Note that the elements in the sets \mathcal{P} and \mathcal{Q} are only finitely additive, not necessarily countably additive. We call the finitely additive set function as charge as in [13] for convenience.

Corollary 3.5 *There exists a charge $Q_\alpha \in \mathcal{Q}$ such that*

$$\sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X] = \sup_{X \in \mathcal{X}_\alpha} E_{\bar{v}}[X]$$

and for any such charge Q_α , $Q_\alpha(\Omega) = 1$.

Proof. Let $\bar{X} := 1 - X$, $\bar{\mathcal{X}}_\alpha := \{\bar{X}; X \in \mathcal{X}_\alpha\}$. Then $\bar{\mathcal{X}}_\alpha$ is also a convex set. From Lemma 3.1, there exists a charge $Q_\alpha \in \mathcal{Q}$ such that

$$\inf_{\bar{X} \in \bar{\mathcal{X}}_\alpha} E_{Q_\alpha}[\bar{X}] = \inf_{\bar{X} \in \bar{\mathcal{X}}_\alpha} E_v[\bar{X}],$$

i.e.

$$1 - \sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X] = 1 - \sup_{X \in \mathcal{X}_\alpha} E_{\bar{v}}[X].$$

Since for any $c \in \mathbb{R}$, we have

$$c = -E_v[-c] = E_{\bar{v}}[c] \leq E_{Q_\alpha}[c] \leq E_v[c] = c.$$

Then $Q_\alpha(\Omega) = E_{Q_\alpha}[1] = 1$. ■

Remark 3.6 *If we consider the problem $\inf_{X \in D} E_\mu[X]$ for some convex set D in \mathcal{X} , by using the same method as in Corollary 3.5, there exists a charge $P_\alpha \in \mathcal{P}$ such that*

$$\inf_{X \in D} E_{P_\alpha}[X] = \inf_{X \in D} E_\mu[X].$$

Lemma 3.7 *If X_α is the optimal solution of Problem 2.2, then for any charge $Q_\alpha \in \mathcal{Q}$ such that*

$$\sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X] = \sup_{X \in \mathcal{X}_\alpha} E_{\bar{v}}[X], \tag{3.1}$$

we have

$$E_{Q_\alpha}[X_\alpha] = \sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X]$$

and $Q_\alpha(\Omega) = 1$.

Proof. We denote the set $\{Q_\alpha \in \mathcal{Q}; \sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X] = \sup_{X \in \mathcal{X}_\alpha} E_{\bar{v}}[X]\}$ as \mathcal{Q}_α . By Corollary 3.5, \mathcal{Q}_α is not empty and any element in it has $Q_\alpha(\Omega) = 1$.

If there exists a charge $Q_\alpha \in \mathcal{Q}_\alpha$ such that

$$E_{Q_\alpha}[X_\alpha] < \sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X]. \quad (3.2)$$

From (3.1) and (3.2), we have

$$E_{\bar{v}}[X_\alpha] \leq E_{Q_\alpha}[X_\alpha] < \sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X] = \sup_{X \in \mathcal{X}_\alpha} E_{\bar{v}}[X].$$

Since $E_{\bar{v}}[X_\alpha] = \sup_{X \in \mathcal{X}_\alpha} E_{\bar{v}}[X]$, we derive contradiction. ■

Definition 3.8 We call a finitely additive set function Q is pure additive, if $Q(\Omega) = 1$ and there exists a sequence of sets $A_n \downarrow \phi$ such that $\lim_{n \rightarrow \infty} Q(A_n) = 1$.

This definition comes from the purely finitely additive set function defined in [17]. By Theorem 1.23 in [17], we know the charge Q_α can be uniquely expressed as

$$Q_\alpha = \lambda Q_\alpha^c + (1 - \lambda) Q_\alpha^p,$$

where Q_α^c is a probability measure, Q_α^p is a pure additive set function and $\lambda \in [0, 1]$.

We need the following two hypotheses:

(H1) For any $A_n \downarrow \phi$ such that $\lim_{n \rightarrow \infty} E_{\bar{v}}[I_{A_n}] \neq 0$, we have $\lim_{n \rightarrow \infty} E_\mu[I_{A_n}] = 0$;

(H2) For any $X \in \mathcal{X}_\alpha$ such that $E_\mu[X] > 0$, we have $E_\mu[X] > E_\mu[(X - \frac{1}{K})^+]$, for any $K \in \mathbb{N}$.

It is easy to check the above hypotheses are obviously true if E_μ is generated by only one bounded countably additive set function. In the rest of the paper, Q_α denotes any charge belong to \mathcal{Q}_α defined in the proof of the Lemma 3.7.

Proposition 3.9 Under (H1) and (H2), if X_α is the optimal solution of Problem 2.2, then

$$E_{(1-\lambda)Q_\alpha^p}[X_\alpha] = 1 - \lambda.$$

Proof. If $E_{(1-\lambda)Q_\alpha^p}[X_\alpha] = \lambda_0 < 1 - \lambda$, there exists a large enough $K \in \mathbb{N}$ such that $\lambda_0 + \frac{1}{K} < 1 - \lambda$. Since Q_α^p is a pure additive set function, there exists a sequence of sets $A_n \downarrow \phi$ such that $\lim_{n \rightarrow \infty} Q_\alpha^p(A_n) = 1$. By (H1) and (H2), there exists a set $A \in \{A_n; n \in \mathbb{N}\}$ such that

$$\lambda_0 + \frac{1}{K} < E_{(1-\lambda)Q_\alpha^p}[I_A] \leq 1 - \lambda$$

and

$$E_\mu[I_A] \leq E_\mu[X_\alpha] - E_\mu[(X_\alpha - \frac{1}{K})^+].$$

Let $X_\alpha^K := (X_\alpha - \frac{1}{K})^+ I_{A^c} + I_A$. We have

$$E_\mu[X_\alpha^K] \leq E_\mu[(X_\alpha - \frac{1}{K})^+ I_{A^c}] + E_\mu[I_A] \leq E_\mu[(X_\alpha - \frac{1}{K})^+] + E_\mu[I_A] \leq E_\mu[X_\alpha] \leq \alpha$$

and

$$\begin{aligned}
E_{Q_\alpha}[X_\alpha^K] &= E_{\lambda Q_\alpha^c}[X_\alpha^K] + E_{(1-\lambda)Q_\alpha^p}[X_\alpha^K] \\
&\geq E_{\lambda Q_\alpha^c}[X_\alpha] - \frac{1}{K} + E_{(1-\lambda)Q_\alpha^p}[I_A] \\
&> E_{\lambda Q_\alpha^c}[X_\alpha] + \lambda_0 = E_{Q_\alpha}[X_\alpha].
\end{aligned}$$

This conflicts with Lemma 3.7. Then $E_{(1-\lambda)Q_\alpha^p}[X_\alpha] = 1 - \lambda$. ■

Lemma 3.10 *Under (H1) and (H2), if X_α is the optimal solution of Problem 2.2, we have*

$$E_{Q_\alpha}[X_\alpha] = \sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X] = \sup_{X \in \mathcal{X}_\alpha} E_{\lambda Q_\alpha^c}[X] + \sup_{X \in \mathcal{X}_\alpha} E_{(1-\lambda)Q_\alpha^p}[X].$$

Proof. The first equation is from Lemma 3.7 and

$$\sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X] \leq \sup_{X \in \mathcal{X}_\alpha} E_{\lambda Q_\alpha^c}[X] + \sup_{X \in \mathcal{X}_\alpha} E_{(1-\lambda)Q_\alpha^p}[X]$$

is obvious.

Denote $\gamma_\alpha^c := \sup_{X \in \mathcal{X}_\alpha} E_{\lambda Q_\alpha^c}[X]$. There exists $\{X_n^c\}_{n \in \mathbb{N}} \subset \mathcal{X}_\alpha$ such that

$$E_{\lambda Q_\alpha^c}[X_n^c] > \gamma_\alpha^c - \frac{1}{n}.$$

By (H1) and (H2), for any $K \in \mathbb{N}$, there exists a set $A_{K,n}$ such that $E_{Q_\alpha^p}[I_{A_{K,n}}] \geq 1 - \frac{1}{K}$ and

$$E_\mu[I_{A_{K,n}}] \leq E_\mu[X_n^c] - E_\mu[(X_n^c - \frac{1}{K})^+].$$

Let $X_n^K := (X_n^c - \frac{1}{K})^+ I_{A_{K,n}^c} + I_{A_{K,n}}$. We have

$$E_\mu[X_n^K] \leq E_\mu[(X_n^c - \frac{1}{K})^+ I_{A_{K,n}^c}] + E_\mu[I_{A_{K,n}}] \leq E_\mu[X_n^c] \leq \alpha$$

and

$$\begin{aligned}
E_{Q_\alpha}[X_n^K] &= E_{\lambda Q_\alpha^c}[X_n^K] + E_{(1-\lambda)Q_\alpha^p}[X_n^K] \\
&\geq E_{\lambda Q_\alpha^c}[X_n^c] - \frac{1}{K} + E_{(1-\lambda)Q_\alpha^p}[I_{A_{K,n}}] \\
&> E_{\lambda Q_\alpha^c}[X_n^c] + 1 - \lambda - \frac{2}{K} \\
&> \gamma_\alpha^c + 1 - \lambda - (\frac{2}{K} + \frac{1}{n}).
\end{aligned}$$

Since K and n can be taken arbitrarily in \mathbb{N} , then

$$\sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X] \geq \gamma_\alpha^c + 1 - \lambda.$$

On the other hand, we have $\sup_{X \in \mathcal{X}_\alpha} E_{(1-\lambda)Q_\alpha^p}[X] \leq 1 - \lambda$. Then

$$\sup_{X \in \mathcal{X}_\alpha} E_{Q_\alpha}[X] \geq \sup_{X \in \mathcal{X}_\alpha} E_{\lambda Q_\alpha^c}[X] + \sup_{X \in \mathcal{X}_\alpha} E_{(1-\lambda)Q_\alpha^p}[X].$$

■

Theorem 3.11 Under (H1) and (H2), if X_α is the optimal solution of Problem 2.2, then it is also the optimal solution of the following problem

$$\sup_{X \in \mathcal{X}_\alpha} E_{\lambda Q_\alpha^c}[X]. \quad (3.3)$$

Proof. By Lemma 3.10, we have

$$E_{\lambda Q_\alpha^c}[X_\alpha] + E_{(1-\lambda)Q_\alpha^p}[X_\alpha] = E_{Q_\alpha}[X_\alpha] = \sup_{X \in \mathcal{X}_\alpha} E_{\lambda Q_\alpha^c}[X] + \sup_{X \in \mathcal{X}_\alpha} E_{(1-\lambda)Q_\alpha^p}[X].$$

Since

$$E_{\lambda Q_\alpha^c}[X_\alpha] \leq \sup_{X \in \mathcal{X}_\alpha} E_{\lambda Q_\alpha^c}[X] \quad \text{and} \quad E_{(1-\lambda)Q_\alpha^p}[X_\alpha] \leq \sup_{X \in \mathcal{X}_\alpha} E_{(1-\lambda)Q_\alpha^p}[X],$$

then

$$E_{\lambda Q_\alpha^c}[X_\alpha] = \sup_{X \in \mathcal{X}_\alpha} E_{\lambda Q_\alpha^c}[X].$$

■

If $\lambda = 0$, i.e. Q_α is a pure additive set function, considering (3.3) becomes meaningless. In this case, since there exists $A_n \downarrow \phi$ such that $\lim_{n \rightarrow \infty} Q_\alpha(A_n) = 1$, i.e. there exists $A_n \downarrow \phi$ such that $Q_\alpha(A_n) = 1$ for any $n \in \mathbb{N}$, under (H1), we can find a large enough N such that A_N satisfying $E_\mu[I_{A_N}] < \alpha$ and $Q_\alpha(A_N) = 1$.

In order to avoid the case $\lambda = 0$ to happen, we give another hypothesis:

(H3) For any sequence $\{A_n\}_{n \in \mathbb{N}}$ such that $A_n \downarrow \phi$, we have $\lim_{n \rightarrow \infty} E_\mu[I_{A_n}] < 1$ and $\lim_{n \rightarrow \infty} E_v[I_{A_n}] < 1$.

In Theorem 3.11, we have proved if X_α is the optimal solution of Problem 2.2, then it is also the optimal solution of (3.3). Thus as long we prove all the optimal solutions of (3.3) has some reminiscent form, then X_α will also has the same reminiscent form.

3.2 The representation of the optimal solution

In this subsection, we will assume (H1)-(H3) hold and focus on solving the problem of (3.3). We will use γ_α^c to denote $\sup_{X \in \mathcal{X}_\alpha} E_{\lambda Q_\alpha^c}[X]$.

3.2.1 The first case

In this subsection, we will give a result for the first case that for all X such that $E_\mu[X] < \alpha$, we have $E_{\lambda Q_\alpha^c}[X] < \gamma_\alpha^c$.

Lemma 3.12 Assume for all $X \in \mathcal{X}_\alpha$ such that $E_\mu[X] < \alpha$, we have $E_{\lambda Q_\alpha^c}[X] < \gamma_\alpha^c$. If X_α is an optimal solution of (3.3), then X_α is also the optimal solution of the following problem:

$$\inf_{Y \in \mathcal{Y}_\alpha} E_\mu[Y],$$

where $\mathcal{Y}_\alpha := \{Y; E_{\lambda Q_\alpha^c}[Y] \geq \gamma_\alpha^c, Y \in [0, 1], Y \in \mathcal{X}\}$.

Proof. Since $E_{\lambda Q_\alpha^c}[Y] < \gamma_\alpha^c$ if $E_\mu[Y] < \alpha$, then for any $Y \in \mathcal{Y}_\alpha$, we have $E_\mu[Y] \geq \alpha$. With $E_\mu[X_\alpha] = \alpha$, the result is proved. ■

We turn to solve (3.3).

Theorem 3.13 *If for all X such that $E_\mu[X] < \alpha$, we have $E_{\lambda Q_\alpha^c}[X] < \gamma_\alpha^c$. Then there exist a real number $\tau \in (0, 1]$ and a probability measure P_α^c such that the optimal solution X_α of (3.3) can be expressed as*

$$X_\alpha = I_{\{H_{\lambda Q_\alpha^c} > \kappa G_{\tau P_\alpha^c}\}} + B I_{\{H_{\lambda Q_\alpha^c} = \kappa G_{\tau P_\alpha^c}\}}, \quad K - a.s.$$

where $H_{\lambda Q_\alpha^c}$ and $G_{\tau P_\alpha^c}$ are the Radon-Nikodym derivatives of λQ_α^c and τP_α^c with respect to $K := \frac{\tau P_\alpha^c + \lambda Q_\alpha^c}{2}$. $\kappa \in \mathbb{R}$ and B is a random variable with values in the interval $[0, 1]$.

Proof. By using the Lemma 3.12, X_α is the optimal solution of the following problem:

$$\inf_{Y \in \mathcal{Y}_\alpha} E_\mu[Y], \quad (3.4)$$

where $\mathcal{Y}_\alpha := \{Y; E_{\lambda Q_\alpha^c}[Y] \geq \gamma_\alpha^c, Y \in [0, 1], Y \in \mathcal{X}\}$.

Then we only need to show that for any optimal solution Y_α of (3.4) has the reminiscent form.

By using the same method as in section 3.1, there exists a charge P_α such that

$$E_{P_\alpha}[Y_\alpha] = \inf_{Y \in \mathcal{Y}_\alpha} E_{P_\alpha}[Y] = \inf_{Y \in \mathcal{Y}_\alpha} E_\mu[Y]$$

and P_α has the unique decomposition

$$P_\alpha = \tau P_\alpha^c + (1 - \tau) P_\alpha^p,$$

where $\tau \in [0, 1]$, P_α^c is a probability measure and P_α^p is a pure additive set function.

Since $E_{\lambda Q_\alpha^c}$ plays the same role as E_μ as in section 3.1 and λQ_α^c is a bounded countable additive set function, then the assumption (H1) and (H2) hold. Under the assumption (H3), we have $\tau \in (0, 1]$. Then the countably part of P_α satisfying

$$E_{\tau P_\alpha^c}[Y_\alpha] = \inf_{Y \in \mathcal{Y}_\alpha} E_{\tau P_\alpha^c}[Y]. \quad (3.5)$$

Since τP_α^c and λQ_α^c are both bounded countable additive set functions, by using the classical Neyman-Pearson lemma (refer the Theorem A.30 in [7]), any optimal solution Y_α of (3.4) has the following form:

$$Y_\alpha = I_{\{\kappa H_{\lambda Q_\alpha^c} > G_{\tau P_\alpha^c}\}} + b \cdot I_{\{\kappa H_{\lambda Q_\alpha^c} = G_{\tau P_\alpha^c}\}}, \quad K - a.s.,$$

where

$$K := \frac{\tau P_\alpha^c + \lambda Q_\alpha^c}{2},$$

$$\kappa := \inf\{u \geq 0; \lambda Q_\alpha^c(u H_{\lambda Q_\alpha^c} \geq G_{\tau P_\alpha^c}) \geq \gamma_\alpha^c\}$$

and B is a suitable random variable taking values in $[0, 1]$.

Since all the optimal solutions of (3.5) have the above form, then any optimal solution of (3.4) has the same form. So does the optimal solution of (3.3). ■

Example 3.14 shows the obtained result is only a necessary condition of the optimal solution.

Example 3.14 *Let $\Omega := [0, 1]$ and \mathcal{F} is the collection of all the Borel sets on $[0, 1]$. $\mathcal{P} := \{P\}$, $\mathcal{Q} := \{\delta_x, x \in [0, 1]\}$, where δ_x is the Dirac measure,*

$$P(\omega) = \begin{cases} \frac{1}{2}, & \omega = \frac{1}{2}; \\ \frac{1}{2}, & \omega = 1. \end{cases}$$

If $\alpha := \frac{1}{2}$, then the optimal solution is

$$X_\alpha = \begin{cases} 1, & \omega \in [0, 1); \\ 0, & \omega = 1. \end{cases}$$

and it is unique.

Every $\delta_x \in \mathcal{Q}$ can be chosen as the Q_α in the Lemma 3.7, but there does not exist a Q_α such that $A = \{H_{Q_\alpha} > \lambda G_P\}$ for some λ , where $A := \{\omega; X_\alpha = 1\}$, H_{Q_α} and G_P are the Radon-Nikodym derivatives with respect to $K := \frac{Q_\alpha + P}{2}$.

Example 3.15 shows the choice of Q_α impacts on finding the optimal solution.

Example 3.15 Let $\Omega := \{\omega_1, \omega_2, \omega_3\}$, \mathcal{F} is all the possible combinations of the elements in Ω . $\mathcal{P} := \{P\}$ and $\mathcal{Q} := \{Q_1, Q_2\}$, where

$$P = \begin{cases} \frac{1}{4}, & \omega = \omega_1; \\ \frac{1}{4}, & \omega = \omega_2; \\ \frac{1}{2}, & \omega = \omega_3, \end{cases} \quad Q_1 = \begin{cases} \frac{1}{2}, & \omega = \omega_1; \\ \frac{1}{2}, & \omega = \omega_2; \\ 0, & \omega = \omega_3 \end{cases} \quad \text{and} \quad Q_2 = \begin{cases} 1, & \omega = \omega_1; \\ 0, & \omega = \omega_2; \\ 0, & \omega = \omega_3. \end{cases}$$

Take $\alpha := \frac{1}{2}$. It is obvious that the optimal solution is $X_\alpha = I_{\{\omega_1\}} + I_{\{\omega_2\}}$. Furthermore, both Q_1 and Q_2 can be considered as the Q_α satisfying the condition in Lemma 3.7.

If we choose Q_2 as the Q_α , $I_{\{\omega_1\}}$ will satisfy

$$E_\mu[I_{\{\omega_1\}}] = \frac{1}{4} < \frac{1}{2}, E_{Q_2}[I_{\{\omega_1\}}] = 1 = \sup_{X \in \mathcal{X}_\alpha} E_{Q_2}[X].$$

3.2.2 The second case

The second case is there exists optimal solution of (3.3) satisfying $E_\mu[X_\alpha] < \alpha$. Example 3.20 shows this case exists.

Proposition 3.16 If an optimal solution X_α^0 of (3.3) satisfying $E_\mu[X_\alpha^0] = \alpha_0 < \alpha$, then for the set $A := \{X_\alpha^0 \neq 1\}$, we have

$$E_{\lambda Q_\alpha^c}[I_A] = 0.$$

Proof. Take

$$X_\alpha^1 := (X_\alpha^0 + \alpha - \alpha_0) \wedge 1.$$

Then $E_\mu[X_\alpha^1] \leq \alpha$ and $E_{\lambda Q_\alpha^c}[X_\alpha^1] \geq E_{\lambda Q_\alpha^c}[X_\alpha^0]$. Since $E_{\lambda Q_\alpha^c}[X_\alpha^0] = E_{\lambda Q_\alpha^c}[X_\alpha^1] = \gamma_\alpha^c$, we have

$$E_{\lambda Q_\alpha^c}[X_\alpha^1 - X_\alpha^0] = 0.$$

Then

$$E_{\lambda Q_\alpha^c}[I_A] = \lim_{\kappa \rightarrow \infty} E_{\lambda Q_\alpha^c}[\kappa(X_\alpha^1 - X_\alpha^0) \bigwedge 1] \leq \lim_{\kappa \rightarrow \infty} E_{\lambda Q_\alpha^c}[\kappa(X_\alpha^1 - X_\alpha^0)] = 0.$$

■

Corollary 3.17 *If there exists an optimal solution X_α^0 of (3.3) satisfying $E_\mu[X_\alpha^0] < \alpha$, then $\gamma_\alpha^c = \lambda$.*

Proof. Since $E_\mu[X_\alpha^0] < \alpha$, from Proposition 3.16, we have the set $A := \{X_\alpha^0 \neq 1\}$ satisfying

$$E_{\lambda Q_\alpha^c}[I_A] = 0,$$

then

$$\gamma_\alpha^c = E_{\lambda Q_\alpha^c}[X_\alpha^0] = E_{\lambda Q_\alpha^c}[I_{A^c}] = \lambda.$$

■

Corollary 3.18 *If there exists an optimal solution X_α^0 of (3.3) satisfying $E_\mu[X_\alpha^0] < \alpha$, for any optimal solution X_α of (3.3), we have $\lambda Q_\alpha^c(A) = 0$, where $A := \{X_\alpha \neq 1\}$.*

Proof. Since X_α is the optimal solution of (3.3), by Corollary 3.17, $E_{\lambda Q_\alpha^c}[X_\alpha] = \gamma_\alpha^c = \lambda$. On the other hand, since $X_\alpha \leq 1$ and $E_{\lambda Q_\alpha^c}[1] = \lambda$, we have $1 - X_\alpha \geq 0$ and $E_{\lambda Q_\alpha^c}[1 - X_\alpha] = 0$. Then $\lambda Q_\alpha^c(A) = 0$. ■

Theorem 3.19 *If there exists an optimal solution X_α^0 of (3.3) satisfying $E_\mu[X_\alpha^0] < \alpha$, then for any probability measure P and any optimal solution X_α of (3.3), X_α can be expressed as*

$$X_\alpha = I_{\{H_{\lambda Q_\alpha^c} > 0\}} + BI_{\{H_{\lambda Q_\alpha^c} = 0\}}, \quad K - a.s..$$

where $H_{\lambda Q_\alpha^c}$ is the Radon-Nikodym derivative of λQ_α^c with respect to $K := \frac{P + \lambda Q_\alpha^c}{2}$ and B is a random variable with values in the interval $[0, 1]$.

Proof. For any optimal solution X_α of (3.3), take $A := \{X_\alpha \neq 1\}$. For any probability measure P , take $K := \frac{P + \lambda Q_\alpha^c}{2}$ and $H_{\lambda Q_\alpha^c} := \frac{d\lambda Q_\alpha^c}{dK}$. From Corollary 3.18, we have $\lambda Q_\alpha^c(A) = 0$. Then $A \subset \{H_{\lambda Q_\alpha^c} = 0\}$ and $\{H_{\lambda Q_\alpha^c} > 0\} \subset A^c = \{X_\alpha = 1\}$. Thus X_α can be expressed as

$$X_\alpha^0 = I_{\{H_{\lambda Q_\alpha^c} > 0\}} + BI_{\{H_{\lambda Q_\alpha^c} = 0\}}, \quad K - a.s.$$

where B is a random variable taking value in $[0, 1]$. ■

Example 3.20 *Let $\Omega := [0, 1]$, \mathcal{F} is all the Borel set on $[0, 1]$, $\mathcal{P} := \{\delta_0\}$ and $\mathcal{Q} := \{\delta_1\}$, where δ_0 and δ_1 are the Dirac measures, i.e.*

$$\delta_0 = \begin{cases} 1, & \omega = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$\delta_1 = \begin{cases} 1, & \omega = 1; \\ 0, & \text{otherwise.} \end{cases}$$

For any given $0 < \alpha < 1$, we have indicator function $I_{\{1\}}$ is always the optimal solution and it obviously has 0-1 structure while $E_\mu[I_{\{1\}}] = E_P[I_{\{1\}}] = 0 < \alpha$. Its representation form is out the framework of [3].

Now we give a necessary and sufficient condition for judging whether there exists an optimal solution X_α of (3.3) while $E_\mu[X_\alpha] < \alpha$.

Theorem 3.21 Denote $\mathcal{B} := \{B \in \mathcal{F}; E_{\bar{\mu}}[I_B] > 0, E_{\lambda Q_\alpha^c}(I_B) = 0\}$ and

$$\beta := \sup_{B \in \mathcal{B}} E_{\bar{\mu}}[I_B].$$

If \mathcal{B} is empty, we define $\beta := 0$.

For any $\alpha \in (0, 1)$, then there exists X_α^0 such that $E_\mu[X_\alpha^0] < \alpha$ and $E_{\lambda Q_\alpha^c}[X_\alpha^0] = \gamma_\alpha^c$ if and only if $\beta > 1 - \alpha$.

Proof. \Leftarrow : If we have $\beta > 1 - \alpha$, there exists a $\hat{B} \in \mathcal{F}$ such that $E_{\bar{\mu}}[I_{\hat{B}}] > 1 - \alpha$ and $E_{\lambda Q_\alpha^c}(I_{\hat{B}}) = 0$. Then $E_\mu[I_{\hat{B}^c}] < \alpha$ and $E_{\lambda Q_\alpha^c}[I_{\hat{B}^c}] = \lambda$, i.e. $I_{\hat{B}^c}$ is a optimal solution of (3.3) satisfying $E_\mu[I_{\hat{B}^c}] < \alpha$.

\Rightarrow : If there exists X_α^0 such that $E_\mu[X_\alpha^0] < \alpha$ and $E_{\lambda Q_\alpha^c}[X_\alpha^0] = \gamma_\alpha^c$, from Corollary 3.17, we know $\gamma_\alpha^c = \lambda$. Let $A_0 := \{X_\alpha^0 \neq 1\}$ and $A_0^c := \{X_\alpha^0 = 1\}$. Since $I_{A_0^c} \leq X_\alpha^0$, we have $E_\mu[I_{A_0^c}] < \alpha$, i.e., $E_{\bar{\mu}}[I_{A_0}] > 1 - \alpha$. Define $\bar{X}_\alpha^0 := 1 - X_\alpha^0$, then $\bar{X}_\alpha^0 > 0$ on set A_0 and $E_{\lambda Q_\alpha^c}[\bar{X}_\alpha^0 I_{A_0}] = E_{\lambda Q_\alpha^c}[1 - X_\alpha^0] = 0$. Thus $E_{\lambda Q_\alpha^c}[I_{A_0}] = 0$. Then $\beta \geq E_{\bar{\mu}}[I_{A_0}] > 1 - \alpha$. ■

By Theorem 3.13 and Theorem 3.19, we have proved that for any X_α , if it is an optimal solution of (3.3), it must have the reminiscent form as in classical case. Since any optimal solution of Problem 2.2 is also the optimal solution of (3.3), we also get the optimal solution of the initial Problem 2.2 must have the reminiscent form as in classical case.

4 The existence of the optimal solution

4.1 A sufficient condition

From the section 3, we know the P_α and Q_α which are chosen by Mazur-Orlicz theorem are crucial important for the representation of the optimal solution. If they are both probability measures, there is no need to decompose them into the countably additive part and the pure additive part and the three assumptions in subsection 3.1 can be abandoned. Now we will give a sufficient condition to guarantee they are both probability measures.

Definition 4.1 We call a sublinear operator ρ continuous from above on \mathcal{X} if for each sequence $\{X_n\}_{n \in \mathbb{N}} \subset \mathcal{X}$ satisfying $X_n \downarrow 0$, we have

$$\rho(X_n) \downarrow 0.$$

Lemma 4.2 If a sublinear operator ρ is continuous from above on \mathcal{X} , then for any linear operator E_P dominated by ρ , P is a probability measure.

Proof. For any $A_n \downarrow \phi$, we have $\rho(I_{A_n}) \downarrow 0$. If a linear operator E_P is dominated by ρ , then $P(A_n) \downarrow 0$. It is easy to see that $P(\Omega) = 1$. Thus, P is a probability measure. ■

Theorem 4.3 If sublinear operators E_μ and E_ν are both continuous from above on \mathcal{X} , then P_α and Q_α in Theorem 3.13 and Lemma 3.7 are probability measures.

Proof. This is the direct result from Lemma 4.2, we omit its proof. ■

Furthermore, if sublinear operators E_μ and E_ν are both continuous from above on \mathcal{X} , the optimal solution of Problem 2.2 exists. To prove the existence of the optimal solution, we need the following result.

Lemma 4.4 *If sublinear operator ρ can be represented by a family of probability measures \mathcal{M} , i.e., $\rho(X) = \sup_{P \in \mathcal{M}} E_P[X]$ and $\bar{\rho}$ is the conjugate operator of ρ , i.e. $\bar{\rho}(X) = \inf_{P \in \mathcal{M}} E_P[X]$. For a sequence $\{X_n\}_{n \in \mathbb{N}}$, if there exists a $M \in \mathbb{R}$ such that $|X_n| \leq M$ for all n , then we have*

$$\rho(\liminf_n X_n) \leq \liminf_n \rho(X_n).$$

and

$$\bar{\rho}(\limsup_n X_n) \geq \limsup_n \bar{\rho}(X_n).$$

Proof. Since the proof of the two results is similar, we only prove the first one. Set $\zeta_n = \inf_{k \geq n} X_k$. Then $\zeta_n \leq X_n$ and $\{\zeta_n\}_{n \in \mathbb{N}}$ is an increasing sequence. It is easy to see that

$$\rho(\liminf_n X_n) = \rho(\lim_n \zeta_n) = \lim_n \rho(\zeta_n) \leq \liminf_n \rho(X_n).$$

■

Theorem 4.5 *If sublinear operators E_μ and E_ν are both continuous from above on \mathcal{X} , then the optimal solution of Problem 2.2 exists.*

Proof. Since $E_\mu[X] = \sup_{P \in \mathcal{P}} E_P[X]$ and $E_\nu[X] = \sup_{Q \in \mathcal{Q}} E_Q[X]$, by the Proposition 2.5 in [9], the elements in \mathcal{P} and \mathcal{Q} are all probability measures, there exists probability measure P_0 such that the elements in \mathcal{P} are all absolutely continuous with respect to P_0 and there exists probability measure Q_0 such that the elements in \mathcal{Q} are all absolutely continuous with respect to Q_0 . Denote $K := \frac{P_0 + Q_0}{2}$. Then any element in \mathcal{P} or \mathcal{Q} is absolutely continuous with respect to K . Take a sequence $\{X_n; n \in \mathbb{N}\} \subset \mathcal{X}_\alpha$ such that

$$E_{\bar{\nu}}[X_n] > \gamma_\alpha - \frac{1}{2^n},$$

where $\gamma_\alpha := \sup_{X \in \mathcal{X}_\alpha} E_{\bar{\nu}}[X]$.

By Komlós theorem, there exists a subsequence $\{X_{n_i}\}_{i \geq 1}$ and a random variable \hat{X} such that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k X_{n_i} = \hat{X}, \quad K - a.s.$$

Since $\{X_n\}_{n \geq 1}$ lies in $[0, 1]$, then \hat{X} lies in $[0, 1]$. By Lemma 4.4, we have

$$E_\mu[\hat{X}] \leq \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k E_\mu[X_{n_i}] \leq \alpha.$$

and

$$E_{\bar{\nu}}[\hat{X}] \geq \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k E_{\bar{\nu}}[X_{n_i}] \geq \lim_{k \rightarrow \infty} (\gamma_\alpha - \frac{1}{k}) = \gamma_\alpha.$$

This shows \hat{X} is the optimal solution of Problem 2.2. ■

4.2 A necessary and sufficient condition

In this section, we will give a necessary and sufficient condition for the existence of the optimal solution of Problem 2.2. Example 4.8 shows that the optimal solution does not always exist.

Definition 4.6 An element $X^* \in \mathcal{X}^*$ is called a subgradient of the function f at X_0 , if

$$\langle X - X_0, X^* \rangle \leq f(X) - f(X_0), \quad \forall X \in \mathcal{X}.$$

The set of all the subgradients of the function f at X_0 is denoted as $\partial f(X_0)$.

Denote $\bar{X} := 1 - X$, then Problem 2.2 can be rewritten as

$$\inf_{\bar{X} \in \bar{\mathcal{X}}_\alpha} E_v[\bar{X}], \quad (4.1)$$

where $\bar{\mathcal{X}}_\alpha := \{\bar{X}; E_{\bar{\mu}}[\bar{X}] \geq 1 - \alpha, \bar{X} \in [0, 1], \bar{X} \in \mathcal{X}\}$.

By the theory of the convex analysis (refer [18]), the dual problem of (4.1) is

$$\max_{\lambda_i} \inf_{\bar{X} \in \bar{\mathcal{X}}} (E_v[\bar{X}] + \lambda_1 g_1(\bar{X}) + \lambda_2 g_2(\bar{X}) + \lambda_3 g_3(\bar{X})), \quad \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0,$$

where $g_1(\bar{X}) := 1 - \alpha - E_{\bar{\mu}}[\bar{X}]$, $g_2(\bar{X}) := \sup_{\omega \in \Omega} \bar{X}(\omega) - 1$ and $g_3(\bar{X}) := -\inf_{\omega \in \Omega} \bar{X}(\omega)$. If we consider the norm as the distance between the elements in \mathcal{X} , then g_1 , g_2 and g_3 are all convex lower semi-continuous functions. Since $\alpha \in (0, 1)$, take $0 < \epsilon < \alpha$ and $\hat{X} := 1 + \epsilon - \alpha$. Then $g_i(\hat{X}) < 0, i = 1, 2, 3$. The Slater condition holds.

Theorem 4.7 \bar{X}_α is the optimal solution of (4.1) if and only if $\bar{X}_\alpha \in \bar{\mathcal{X}}_\alpha$ and there exist nonnegative real numbers $\bar{\lambda}_i, i = 1, 2, 3$, such that $\bar{\lambda}_1 g_1(\bar{X}_\alpha) = 0, \bar{\lambda}_2 g_2(\bar{X}_\alpha) = 0, \bar{\lambda}_3 g_3(\bar{X}_\alpha) = 0$ and

$$0 \in \partial E_v[\bar{X}_\alpha] + \bar{\lambda}_1 \partial g_1(\bar{X}_\alpha) + \bar{\lambda}_2 \partial g_2(\bar{X}_\alpha) + \bar{\lambda}_3 \partial g_3(\bar{X}_\alpha).$$

Proof. Refer the Theorem 2.9.2 and 2.9.3 in [18]. ■

Example 4.8 Consider $\Omega := \mathbb{N}$, $\mathcal{F} := 2^\Omega$, $E_\mu(X) := E_{P_1}[X]$ and $E_v[X] := E_{P_2}[X]$, where P_1 is a pure finitely additive set function taking values only among $\{0, 1\}$ which gives 0 to singletons and 1 to \mathbb{N} , $P_2(k) = \frac{1}{2^k}, k \in \mathbb{N}$.

Let $A_n := \{1, 2, \dots, n\}$. For any significance level $\alpha \in (0, 1)$, we have $E_{P_1}[I_{A_n}] = 0$ and $E_{P_2}[I_{A_n}] = 1 - \frac{1}{2^n}$. Then

$$\sup_{X \in \mathcal{X}_\alpha} E_{P_2}[X] = 1.$$

If there exists a \hat{X} in \mathcal{X}_α such that $E_{P_2}[\hat{X}] = 1$, then $X \equiv 1$. This contradicts with $E_{P_1}[\hat{X}] = 1 \leq \alpha$. Thus the optimal solution in such case does not exist.

5 When the tests are in L_c^1 -space

Some theories consider a small space in place of the whole bounded measurable functions. For example, the G -expectation introduced by Peng [12] which considered the L_c^1 -space is in such case. In this section, we will study the hypothesis testing problem when the test functions are restrained to be chosen in L_c^1 space. Firstly, we give the definition of the L_c^1 -space, which comes from [5].

Let Ω be a complete separable metric space equipped with the distance d and $\mathcal{B}(\Omega)$ the Borel σ -field of Ω . $C_b(\Omega)$ is all continuous bounded $\mathcal{B}(\Omega)$ -measurable real functions.

For two sublinear operators E_μ and E_v , we take $\rho(X) := E_\mu[X] \vee E_v[X]$. It is easy to check ρ is a sublinear operator. Let us denote

$$c(A) := \rho(I_A)$$

Definition 5.1 *The set A is polar if $c(A) = 0$ and we call a property holds "quasi-surely"(q.s.) if it holds outside a polar set.*

Furthermore, we denote

$$\begin{aligned}\mathcal{L}^1 &= \{X \in \mathcal{X}; \rho(|X|^1) < \infty\}, \\ \mathcal{N} &:= \{X \in \mathcal{X}; X = 0, \quad c - q.s.\}, \\ L^1 &:= \mathcal{L}^1 / \mathcal{N},\end{aligned}$$

and L^1 is a Banach space with the norm $\|X\|_{L^1} := \rho(|X|)$

L_c^1 is the completeness of the $C_b(\Omega)$ under the L^1 -norm.

The hypothesis testing problem is:

Problem 5.2 *For two sublinear operators E_μ and E_v , whether there exists a $f_\alpha \in L_c^1$ such that*

$$E_{\bar{v}}[f_\alpha] = \sup_{f \in L_{c,[0,1]}^{1,\alpha}(\Omega)} E_{\bar{v}} E_Q[f], \quad (5.1)$$

where α is the significance level lies in $(0, 1)$ and $L_{c,[0,1]}^{1,\alpha} := \{f \in L_c^1; 0 \leq f \leq 1, E_\mu[f] \leq \alpha\}$.

Definition 5.3 *We call a sublinear operator ρ continuous from above on L_c^1 if for each sequence $\{f_n\}_{n \in \mathbb{N}} \subset L_c^1$. satisfying $f_n \downarrow 0$, c-q.s., we have*

$$\rho(f_n) \downarrow 0.$$

By the Corollary 33 in [5], we know a sublinear operator ρ is continuous from above on L_c^1 if there exists a probability measure set \mathcal{M} which is weakly compact such that $\rho(X) = \sup_{P \in \mathcal{M}} E_P[X]$.

Theorem 5.4 *If the sublinear operators E_μ and E_v are both continuous from above on L_c^1 and the optimal solution of Problem 5.2 exists, then any optimal solution of Problem 5.2 has the reminiscent form as in section 4.*

Proof. The whole proof is similar as in section 3, we only point out that since E_μ and E_ν are both continuous from above on L_c^1 , then we can do as in subsection 4.1 and get P_α and Q_α chosen as in subsection 3 by Mazur-Orlicz theorem satisfying: for any $\{f_n\}_{n \in \mathbb{N}} \subset L_c^1$ satisfying $f_n \downarrow 0$, c-q.s., we have $E_{P_\alpha}(f_n) \downarrow 0$ and $E_{Q_\alpha}(f_n) \downarrow 0$. By the Daniell-Stone theorem, P_α and Q_α can be chosen both as probability measures. The rest proof is similar. We omit it. ■

Note that since P_α and Q_α can be chosen both as probability measures in this case, then the three assumptions in subsection 3.1 can be abandoned.

The last example is about G -expectation.

Peng introduces an sublinear expectation which is called G -Expectation in [12]. In [5], Denis Hu and Peng give a specific represent form for G -expectation. Now under the frame of [12], we give an example for our problem.

Example 5.5 Let $(\Omega, \mathcal{F}, P_0)$ be a probability space and $(W_t)_{t \geq 0}$ be the 1-dimensional Brownian motion in this space. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the augmented σ -algebra generated by $W(\cdot)$. Let Θ be a given bounded and closed subset in R . Denote $\mathcal{A}_{t,T}^\Theta$ as the collection of all Θ -valued \mathbb{F} -adapted process on an interval $[t, T] \subset [0, \infty)$. For each fixed $\theta \in \mathcal{A}_{t,T}^\Theta$, denote

$$B_t^{0,\theta} := \int_0^t \theta_s dW_s.$$

Define $P_\theta := P_0 \circ (B^{0,\theta})^{-1}$, then the G -expectation $\mathbb{E}[\cdot]$ introduced by Peng can be written as

$$\mathbb{E}[\phi(B_{t_1}^0, B_{t_2}^{t_1}, \dots, B_{t_n}^{t_{n-1}})] = \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{P_\theta}[\phi(B_{t_1}^0, B_{t_2}^{t_1}, \dots, B_{t_n}^{t_{n-1}})]$$

Given two families of probability measure $\mathcal{P} := \{P_\theta, \theta \in \Theta_1\}$ and $\mathcal{Q} := \{Q_\theta, \theta \in \Theta_2\}$, $\Theta_1 \cap \Theta_2 = \emptyset$. It is easy to check that $E_\mu[I_{\{\langle B \rangle_1 \in \Theta_2\}}] = 0$ and $E_\nu[I_{\{\langle B \rangle_1 \in \Theta_2\}}] = 1$, where $\langle B \rangle$ is the quadratic variation process of B . If we consider discriminating two G -expectation $E_\mu[\cdot]$ and $E_\nu[\cdot]$, i.e. to discriminate Θ_1 and Θ_2 , then for any significance level α , $I_{\{\langle B \rangle_1 \in \Theta_2\}}$ is the optimal solution of Problem 2.2.

References

- [1] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. (1999) Coherent measures of risk. Mathematical Finance, 9(3), 203-228.
- [2] Basile, A. and Rao, K, P, S, B. (2000) Completeness of L_p -Spaces in the finitely additive setting and related stories, Journal of Mathematical Analysis and Applications 248, 588-624.
- [3] Cvitanic, J. and Karatzas, I. (2001) Generalized Neyman-Pearson lemma via convex duality. Bernoulli, 7, 79-97.
- [4] Deimling, K. (1985) Nonlinear Functional Analysis. Springer-Verlag, New York/Berlin.

- [5] Denis, L. Hu, M. and Peng, S. (2011), Function spaces and capacity related to a sublinear expectation: Application to G -Brownian motion paths, *Potential Anal.* 34, 139-161.
- [6] Dunford, N. and Schwartz, J.T. (1958), *Linear Operators, Part I*, Wiley-Interscience/New York.
- [7] Föllmer, H. and Schied, A. (2002) *Stochastic Finance. An introduction in discrete time*. Walter de Gruyter, Berlin/New York.
- [8] Huber, P. and Strassen, V. (1973) Minimax tests and the Neyman-Pearson lemma for capacities. *The annals of Statistics*, 1, 251-263.
- [9] Ji, S. and Sun, C. (2014) The minimum mean square estimator for a sublinear operator. arXiv:1412.5736.
- [10] Ji, S. and Zhou, X. (2010) A generalized Neyman-Pearson lemma for g -probabilities. *Probab. Theory Relat. Fields*, 148, 645-669.
- [11] Komlós, J. (1967) A generalization of a problem of Steinhaus. *Acta Math. Acad. Sci. Hungar.*, 18, 217-229.
- [12] Peng, S. (2007) G -expectation, G -Brownian motion and related stochastic calculus of Itô type. *Stochastic Analysis and Applications*, 2, 541-567.
- [13] Rao, K, P, S, B. and Rao, M, B. (1983), *Theory of Charge*, Academic Press Inc/London.
- [14] Rudloff, B. (2007). Convex hedging in incomplete markets. *Appl. Math. Finance*, 14, 437-452.
- [15] Rudloff, B. and Karatzas, I. (2010) Testing composite hypotheses via convex duality. *Bernoulli*, 16, 1224-1239.
- [16] Simons. S. (2008) *From Hahn-Banach to Monotonicity*. Springer-Verlag, Berlin/Heidelberg.
- [17] Yosida, K. and Hewitt, E. (1952), Finitely additive measures, *Trans. Amer. Math. Soc*, Vol. 72, No. 1. 46-66.
- [18] Zălinescu, C. (2002), *Convex Analysis in General Vector Spaces*, World Scientific, River Edge, NJ.